THE BEHAVIOURAL APPROACH to SYSTEMS and CONTROL

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OUTLINE

- **Introduction**
- **The behavior**
- **Modeling interconnected systems**
- **Elimination**
- **Controllability**
- **Representations**
- **Algorithmic issues**
- **Control as interconnection**

SALIENT FEATURES

- **Dynamical system =**
  a behavior

  input/output structure =
  important special case

- **First principles models**
  $\Rightarrow$ latent variables

  state variables =
  important special case

- **Modeling complex systems**
  = tearing & zooming

  i/o cascade and feedback =
  limited special case

- **Control = interconnection**

  feedback =
  important special case

SYSTEM

Let $w_1, w_2, \cdots, w_q$ be variables whose dynamic relation we wish to describe (modeling), or analyze, or design (control and synthesis)

....

Ingredients:

'Time set' $\mathbb{T}$

(today, $\mathbb{T} = \mathbb{R}$ but framework covers $\mathbb{Z}$, DES, hybrid systems)

w_k takes on its value in $\mathbb{W}_k$.

Yields $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2 \times \cdots \times \mathbb{W}_k$

'signal space'

How do we express the 'laws'?

SYSTEM (continued)

Classical framework:

some of w_k's act as
inputs, causes, stimuli,
the other w_k's act as
outputs, effects, responses.

system = I/O map
transfer function, etc.

internal initial conditions
→ familiar I/S/O representation

$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u)$$

via output to input cascade + feedback interconnection : complex systems

very successful framework in signal processing, control, simulation, etc.

SYSTEM (continued)

Limitations:

- cause/effect, stimulus/response is often simply not physical!
- very awkward framework for use in first principles modeling: what is the signal flow graph?
- not well-suited for computer assisted ‘object oriented’ modeling!!!
- unnecessarily limits model representations.

THE BEHAVIOR

The *behavioral approach* takes the feasible trajectories

$$w : \mathbb{T} \to \mathbb{W}$$

$$w = (w_1, w_2, \cdots, w_q)$$

as the object of study.

Totality of feasible trajectories = $\mathfrak{B}$
the *'behavior'*; $\mathfrak{B} \subset \mathbb{W}^{T}$

$w \in \mathfrak{B}$ **: the system allows the trajectory** $w = (w_1, w_2, \cdots, w_q)$

$w \notin \mathfrak{B}$ **: the system forbids the trajectory** $w = (w_1, w_2, \cdots, w_q)$

Note: in I/O-systems
$\mathfrak{B}$ **= all** (u, y)**-pairs.**

Note: analogy with formal language: some words OK, some words not OK.

THE BEHAVIOR (continued)

'Word' examples

- Planetary orbits
- Port behavior of an electrical circuit

- Force/position behavior in mechanical systems

- other frameworks, that have yielded away from I/O modeling:
  across/through variables
  (bondgraphs)
  intensive/extensive variables
  (thermodynamics)
  formal languages, PDE's
  (cs, physics)

BEHAVIORAL EQUATIONS

Differential equations
$(\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{R}^{\mathtt{w}})$,

$$F(w(t), \frac{dw}{dt}(t), \cdots, \frac{d^n w}{dt^n}(t)) = 0$$

model $\cong F$.

Linear case

$$R_0 w + R_1 \frac{dw}{dt}(t) + \cdots + R_n \frac{d^n w}{dt^n}(t) = 0$$

$R_0, R_1, \cdots, R_n$: constant matrices = model parameters.

behavior = all solutions

BUT! Models are seldomly given this way!

Auxiliary variables !!!

LATENT VARIABLES

Behavioral equations:

$$F(w(t), \frac{dw}{dt}(t), \cdots, \frac{d^n w}{dt^n}(t), \ell(t), \frac{d\ell}{dt}(t), \cdots, \frac{d^n \ell}{dt^n}(t)) = 0$$

Behavior: all 'solutions' $w : \mathbb{R} \to \mathbb{W}$ that is, all $w : \mathbb{R} \to \mathbb{W}$, for which there exists $\ell : \mathbb{R} \to \mathbb{L}$, such that (w, ℓ) is a solution.

w: 'manifest' variables
ℓ: 'latent' variables

Example:

$$\frac{d}{dt}x = f(x,u), y = h(x,u)$$
$$w = (u,y)$$

x: latent, (u,y): manifest.

Latent $\cong$ internal,
but internal $\neq$ state.

INTERCONNECTED SYSTEM

Graph with leaves

nodes = modules

edges = connections

= pairing of terminals

leaves = external terminals

Think about an electrical circuit

MODELING

A computer-assistance oriented procedure for obtaining a model for an interconnected system

Central notions involved:

- **terminals**
- **modules**
- **interconnection architecture**

....

modules	=	**building blocks**
terminals	=	**links between subsystems carry variables that vary with time**
interconnection architecture	=	**the way the subsystems are linked**

TERMINALS

A *terminal* is specified by its *type*.

The *type* implies an ordered set of *variables*.

Examples

Type of terminal	**Variables**
electrical	**(voltage, current)**
mechanical (1-D)	**(force, position)**
mechanical (2-D)	**(force, torque, position, attitude)**
thermal	**(temp, heat flow)**
fluidic	**(pressure, flow)**
m-dim input	$(u_1, u_2, \cdots, u_m)$
p-dim output	$(y_1, y_2, \cdots, y_p)$
etc.	**etc.**

MODULES

A ***module*** is specified by ***its type***, ***its representation*** and its ***parameter values.***

The type specifies an ordered set of terminals $(t_1, t_2, \cdots, t_n)$

This specifies an ordered set of variables $(w_1, w_2, \cdots, w_n)$

The ***representation*** specifies the behavior of these variables.

This representation contains parameters. The ***parameter values*** specify their values.

By specifying a module we obtain the behavior of the variables
$(w_1, w_2, \cdots, w_n)$
on the terminals of the module.

MODULES (continued)

Examples

Module	Repr.	Par. value
resistor	imp	$\boldsymbol{R}$ in ohms
resistor	adm	$\boldsymbol{G}$ in mhos
cap	default	$\boldsymbol{C}$ in farad
$\triangle$	default	$\boldsymbol{R}$ in ohms
transformer	default	turns ratio
m-port Imp	tf. fn.	$\boldsymbol{G} \in \mathbb{R}^{m \times m}(\boldsymbol{\xi})$
m-port imp	state repr.	$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$
m-port imp	kernel repr.	$\boldsymbol{R} \in \mathbb{R}^{m \times 2m}[\boldsymbol{\xi}]$
m-port imp	im repr.	$\boldsymbol{M} \in \mathbb{R}^{2m \times m}[\boldsymbol{\xi}]$
mass	default	$\boldsymbol{m}$ in kgr
pendulum	default	$\boldsymbol{m}$ and $\boldsymbol{L}$
2 inlet tank	default	geometry
(m,p) lin sys	state repr.	(A,B,C,D)
etc.	etc.	etc.

MODULES (continued)

terminals = (t_1, t_2)
t_1, t_2 : both electrical
$t_1 \rightarrow (V_1, I_1),\ t_2 \rightarrow (V_2, I_2)$
behavioral equations:

$$V_1 - V - 2 = RI_1,\ I_1 = I_2$$

....
terminals=(t_1, t_2)
behavioral equations

$$C\frac{d}{dt}(V_1 - V_2) = I_1,\ I_1 = I_2$$

....
terminals=(t_1, t_2, t_3)
latent variables: I_1', I_2', I_3'
behavioral equations:

$$V_1 - V_2 = RI_3', V_2 - V_3 = RI_1', V_3 - V_1 = RI_2'$$
$$I_1 = I_3' - I_2', I_2 = I_1' - I_3', I_3 = I_2' - I_1'$$

MODULES (continued)

terminals=$(t_1, t_2, \cdots, t_m$,ground)
all electrical
latent variables: x
behavioral equations:

$$\frac{d}{dt}x = Ax + BI',\ V' = Cx + DI'$$

$$(I_1, \cdots, I_m, I_{m+1}) = (I'_1, \cdots, I'_m, -I'_1 - I'_2 \cdots - I'_m)$$

$$(V_1, \cdots, V_m, V_{m+1}) = (V'_1 + V, \cdots, V'_m + V, V)$$

INTERCONNECTION ARCHITECTURE

The interconnection architecture is a list of *terminal pairs*

Connect	
t'	t''
t'''	t''''
etc.	etc.
etc.	etc.

$\{t', t''\}$ can be such a pair only if the type of t' is suitably adapted to the type of t''

"adapted"

$\rightarrow$ same type (electrical, mechanical, thermal)

$\rightarrow$ output to input for 'logical' connections.

INTERCONNECTION ARCHITECTURE (continued)

The interconnection architecture imposes restrictions on the variables 'living' on the associated terminals.

Electrical: $t' \to V', I', \;\; t' \to V'', I''$
restriction: $V' = V'', \;\; I' + I'' = 0$

1-D mechanical:
$t' \to F', q', \;\; t'' \to F'', q''$
restriction: $F' + F'' = 0, \;\; q' = q''$

2-D mechanical:
$t' \to x', y', \theta', X', Y', T'$
$t'' \to x'', y'', \theta'', X'', Y'', T''$
restriction: $x' = x'', y' = y'', \theta' = -\theta'', \; X' + X'' = 0, \;\; Y' + Y'' = 0 \;\; T' = T''$

thermal $t' \to T', Q', \;\; t' \to T'', Q''$
restriction: $T' = T'', \;\; Q' + Q'' = 0$

logical $t' \to u', \;\; t' \to y''$
restriction: y=u

MODEL GENERATION

So in order to obtain a model specify:

- **Modules $M_1, M_2, \cdots, M_N$**
  type + representation +
  parameter values.
  This yields a list of terminals $t_1, t_2, \cdots, t_N$ and a behavior $\mathfrak{B}'$ for the variables living on the terminals.
- **Interconnection architecture on $t_1, t_2, \cdots, t_N$**
  this yields a behavior $\mathfrak{B}''$ for the variables living on the terminals
- **$\mathfrak{B}' \bigcap \mathfrak{B}''$ = the behavior of the interconnected system**
  contains latent variables and manifest variables.
- **Elimination of latent variables $\rightarrow$ $\mathfrak{B}$**

I/O and INTERCONNECTIONS

Consider 2 tanks:

$\frac{d}{dt}h_1 = F_1(h_1, p_1, p_2)$	$\frac{d}{dt}h_2 = F_3(h_3, p_3, p_4)$
$f_1 = H_1(h_1, p_1)$	$f_3 = H_3(h_3, p_3)$
$f_2 = H_2(h_1, p_2)$	$f_4 = H_4(h_3, p_4)$
input: p_1, p_2	input: p_3, p_4
output: f_1, f_2	output: f_3, f_4

Interconnection: $p_2 = p_3, f_2 + f_3 = 0$

input=input; output=output

very many such examples (e.g., in mechanics, heat transfer)

RLC circuit

! **Model the relation between V_1, I_1, V_2, I_2.**

RLC circuit (continued)

Module	type	terminals	par. value
I	connector	1,3,4	
II	capacitor	5,6	C
III	resistor	7,8	R_C
IV	resistor	9,10	R_L
V	inductor	11,12	L
VI	connector	13,14,2	

Architecture: connect terminals

3	5
4	7
6	9
8	11
10	13
12	14

RLC circuit (continued)

Each terminal has two real variables:
voltage V, current I.
Generate the behavioral equations via module laws & interconnections:

Modules	
$I_1 + I_3 + I_4 = 0$	$V_1 = V_3 = V_4$
$I_2 + I_{13} + I_{14} = 0$	$V_2 = V_{13} = V_{14}$
$I_5 + I_6 = 0$	$C\frac{d}{dt}(V_5 - V_6) = I_5$
$I_7 + I_8 = 0$	$V_7 - V_8 = R_C I_7$
$I_9 + I_{10} = 0$	$V_9 - V_{10} = R_L I_9$
$I_{11} + I_{12} = 0$	$V_{11} - V_{12} = L\frac{d}{dt}I_{11}$

Interconnections	
$V_3 = V_5$	$I_3 = -I_5$
$V_4 = V_7$	$I_4 = -I_7$
$V_6 = V_9$	$I_6 = -I_9$
$V_8 = V_{11}$	$I_8 = -I_{11}$
$V_{10} = V_{13}$	$I_{10} = -I_{13}$
$V_{12} = V_{14}$	$I_{12} = -I_{14}$

Model for V_1, I_1, V_2, I_2: manifest,
$V_3, I_3, \cdots, V_{14}, I_{14}$: latent.

RLC circuit (continued)

! Eliminate $V_3, I_3, \cdots, V_{14}, I_{14}$!

After suitable manipulations, we obtain:

CART with DOUBLE PENDULUM

! Model relation between u and y.

DOUBLE PENDULUM CART (continued)

View as interconnection of 5 modules

M_1 : pointmass with 2 rigid pins and 1 hinge
terminals (t_1, t_2, t_3)
2−D mechanical

M_2 : pendulum with rigid bar and 1 hinge
terminals (t_4, t_5)
2−D mechanical

M_3 : pendulum with rigid base
terminal (t_6)
2−D mechanical

M_4 : horizontal drive
terminal (t_7)
2−D mechanical

M_5 : horizontal free motion pin
terminal (t_8)
2-D mechanical

DOUBLE PENDULUM CART (continued)

Interconnection architecture

Connect	
t_1	t_7
t_2	t_8
t_3	t_4
t_5	t_6

DOUBLE PENDULUM CART (continued)

Description of modules

M_1: pointmass with 2 rigid pins and 1 hinge
parameter values <u>mass</u>, $\underline{\Delta\theta}$

variables:

$$x_1,\ y_1,\ \theta_1,\ X_1,\ Y_1,\ T_1$$
$$x_2,\ y_2,\ \theta_2,\ X_2,\ Y_2,\ T_2$$
$$x_3,\ y_3,\ \theta_3,\ X_3,\ Y_3,\ T_3$$

latent variables:

$$x,\ y,\ \theta,\ X,\ Y$$

DOUBLE PENDULUM CART (continued)

Behavioral equations:

$$
\begin{aligned}
x_1 &= x_2 = x_3 = x \\
y_1 &= y_2 = y_3 = y \\
\theta_2 &= \theta_1 + \underline{\Delta\theta} \\
X &= X_1 + X_2 + X_3 \\
Y &= Y_1 + Y_2 + Y_4 \\
\underline{\text{mass}}\, \frac{d^2x}{dt^2} &= X \\
\underline{\text{mass}}\, \frac{d^2y}{dt^2} &= Y \\
T_1 + T_2 &= 0 \\
T_3 &= 0
\end{aligned}
$$

DOUBLE PENDULUM CART (continued)

M_2 : pendulum with rigid base and one hinge
parameter values <u>length</u>, <u>mass</u>

variables:

$$\boldsymbol{x_1,\ y_1,\ \theta_1,\ X_1,\ Y_1,\ T_1}$$
$$\boldsymbol{x_2,\ y_2,\ \theta_2,\ X_2,\ Y_2,\ T_2}$$

latent variables:

$$\boldsymbol{x,\ y,\ \theta,\ X,\ Y}$$

DOUBLE PENDULUM CART (continued)

Behavioral equations:

$$
\begin{aligned}
x_1 &= x - \underline{\text{length}} \cos \theta_1 \\
x_2 &= x \\
y_1 &= y - \underline{\text{length}} \sin \theta_1 \\
y_2 &= y \\
X &= X_1 + X_2 \\
Y &= Y_1 + Y_2 \\
\underline{\text{mass}}\ \frac{d^2 x}{dt^2} &= X \\
\underline{\text{mass}}\ \frac{d^2 y}{dt^2} &= Y - \underline{\text{mass}}\ g \\
\underline{\text{mass}}\ (\underline{\text{length}})^2\ \frac{d^2 \theta}{dt^2} &= T_1 - \underline{\text{mass}}\ g\ \underline{\text{length}}\ \cos \theta_1 \\
T_2 &= 0
\end{aligned}
$$

DOUBLE PENDULUM CART (continued)

M_3: Pendulum with rigid base
parameter values: <u>length</u>, <u>mass</u>

variables:

$$x,\ y,\ \theta,\ X,\ Y,\ T,$$

behavioral equations:

$$\begin{aligned}
\underline{\text{mass}}\ \frac{d^2x}{dt^2} &= X \\
\underline{\text{mass}}\ \frac{d^2y}{dt^2} &= Y - \underline{\text{mass}}\ g \\
\underline{\text{mass}}\ (\underline{\text{length}})^2\ \frac{d^2\theta}{dt^2} &= T - \underline{\text{mass}}\ g\ \underline{\text{length}}\ cos\theta
\end{aligned}$$

DOUBLE PENDULUM CART (continued)

M_4: horizontal drive

external force input $u : \mathbb{R} \rightarrow \mathbb{R}$

variables:

$$x, y, \theta, X, Y, T$$

behavioral equations:

$$\begin{aligned} y &= 0 \\ \theta &= 0 \\ X &= u \\ Y &= 0 \\ T &= 0 \end{aligned}$$

DOUBLE PENDULUM CART (continued)

M_5: horizontal free motion

variables:

$$x,\ y,\ \theta,\ X,\ Y,\ T$$

behavioral equations:

$$\begin{gathered} y = 0 \\ \theta = -\frac{\pi}{2} \\ X = 0 \\ T = 0 \end{gathered}$$

DOUBLE PENDULUM CART (continued)

Write the interconnection laws:

For each interconnection, the 2-D mechanical interconnection laws:

$$\begin{aligned} x' &= x'' \\ y' &= y'' \\ \theta' &= -\theta'' \\ X' &= -X'' \\ Y' &= -Y'' \\ T' &= T'' \end{aligned}$$

etc., etc.

DOUBLE PENDULUM CART (continued)

After suitable manipulations, we obtain:

ELIMINATION

Consider

$$\boldsymbol{R}(\frac{d}{dt})\boldsymbol{w} = \boldsymbol{M}(\frac{d}{dt})\boldsymbol{\ell} \qquad (\star)$$

$\boldsymbol{w}$: $\mathtt{w}$-dimensional, $\boldsymbol{\ell}$: $\mathtt{l}$-dimensional

$\boldsymbol{R} \in \mathbb{R}^{\bullet\times\mathtt{w}}[\boldsymbol{\xi}]$, $\boldsymbol{M} \in \mathbb{R}^{\bullet\times\mathtt{l}}[\boldsymbol{\xi}]$
$\boldsymbol{R}(\boldsymbol{\xi}) = \boldsymbol{R}_0 + \boldsymbol{R}_1\boldsymbol{\xi} \cdots \boldsymbol{R}_n\boldsymbol{\xi}^n$
$\boldsymbol{M}(\boldsymbol{\xi}) = \boldsymbol{M}_0 + \boldsymbol{M}_1\boldsymbol{\xi} \cdots \boldsymbol{M}_n\boldsymbol{\xi}^n$

....

Define

$$\mathfrak{B}_{\text{full}} = \left\{\boldsymbol{w} : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{l}} \mid (\star) \text{ holds}\right\}$$

$$\mathfrak{B} = \{w \mid \text{there is an } \ell \text{ such that } (\star) \text{ holds}\}$$

ELIMINATION (continued)

Questions:

- **Is $\mathfrak{B}$ described by a constant-coefficient differential equation?**

  i.e., is there $R' \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

  $$R'(\frac{d}{dt}) = 0 \quad R' \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$$

  is a kernel representation of $\mathfrak{B}$?

- **If so, find and algorithm**

  $$(R, M) \mapsto R'$$

- **Nonlinear, PDE generalizations?**

ELIMINATION (continued)

Theorem:
There indeed exists an R' such that

$\mathfrak{B}$ is the solution set of

$$R'(\frac{d}{dt})w = 0$$

....
Projection of linear differential behavior = same!

....

Algorithm for computing R':

$n \in \mathbb{R}^{1\times\bullet}$ is *annihilator* for M if $nM = 0$.
Set of annihilators = module; finitely generated
Generators = $(n_1, n_2, \cdots, n_g)$; defines matrix N
$R' = NR$.
To be made into computer algebra

Pseudocode for elimination

```
function R'=LATELIM(R,M);
% Input :  Polynomial matrices M and R with same number of rows
% Output:  Polynomial matrix R' such that R'(d/dt)w = 0 is the manifest
%          behavior corresponding to the hybrid representation
% R(d/dt)w = M(d/dt)ℓ
%
% At each step, the degree of one of thew rows of M is decreased by adding a
% suitable polynomial combination of other rows.  The same operation
% is applied to the rows of R.  When a row of M becomes 0, the
% corresponding row of R becomes part of R'.
```

```
[M, R]=Order(M, R);
 % order rows so that rows of M
are in decreasing degree order
i =rowdim(M); p = i;
k = 1;

while(all(M(i,:) == 0)) %check
if M has zero rows
    R'[k,:] = R[i,:]; %corresponding
rows of R are in K
    i = i - 1;   p = p - 1;   k =
k + 1;
end

while ((i ≥ 1)) %reduction procedure


    if (∃ real n s.t.  M_hc(i,:) =
n M_hc(i + 1 : p,:)) then
     % check if highest coefficient
is linearly dependent
     % from those of rows of lower
or equal degree

        h=polann(n, M); %polynomial
vector to reduce degree
        m = M(i,:) - hM(i + 1 :
p,:); % degree(m)< degree(M(i,:))
        r = R(i,:) - hR(i+1 : p,:
); %same combination on R
```

```
            [M, R]=Eliminate(M, R, i);
% eliminate row
            if (m ≠ 0) then
               [M, R, j]=Insert(M, R, m, r);
               % insert m and r as j-th
row of M and R
               % so as to keep rows of
M ordered by degree
               i = j;
            else % the combination of
rows of M is zero
               R'[k, :] = r %combination
of rows of R in K
               i = i - 1;   p = p -
1;   k = k + 1;
            end
         else i = i - 1;
         end
      end
```

CONTROLLABILITY

The time-invariant system $(\mathbb{R}, \mathbb{W}, \mathfrak{B})$ is siad to be

controllable

if for all $w_1 \in \mathfrak{B}, w_2 \in \mathfrak{B}$,

there exists $t' \geq 0$ and $w \in \mathfrak{B}$

such that

$$w(t) = \begin{cases} w_1(t) & for \;\; t < 0 \\ w_2(t - t') & for \;\; t \geq t' \end{cases}$$

CONTROLLABILITY (continued)

Questions:

- Is the system described by

$$R(\frac{d}{dt})w = 0$$

  controllable?

- Find an effective algorithm for verifying controllability in terms of $R_0, R_1, ..., R_n$ where

$$R(\xi) = R_0 + R_1\xi + \cdots + R_n\xi^n)$$

- Nonlinear, PDE generalizations?

- Application/relevance in control

CONTROLLABILITY (continued)

Theorem:

The following are equivalent

1. $R(\frac{d}{dt})w = 0$ **defines a controllable system**

2. **rank**$(R(\lambda))$ **is independent of** λ **for** $\lambda \in \mathbb{C}$

3. **The behavior** $\mathfrak{B}$ **is the manifest behavior of**

$$w = M(\frac{d}{dt})\ell$$

for some $M \in \mathbb{R}^{\mathtt{w}\times\bullet}$

$\mathfrak{B}$ **admits an** ***"image representation"*****.**

CONTROLLABILITY Verification

Idea

Given R

> **$f \in \mathbb{R}^{\bullet\times 1}[\xi]$ belongs to the SYZYGY of R if $Rf = 0$**

SYZYGY = module, finitely gen.
generators $\{r'_1, r'_2, \cdots, r'_g\}$
form matrix R'

annihilators of $R =: \mathcal{N}_R$
annihilators of $R' =: \mathcal{N}_{R'}$

Controllability test:

$$\mathcal{N}_R = \mathcal{N}_{R'}?$$

$\rightarrow$ Computer algebra

Pseudocode for verifying controllability

```
function [M,obs]=RPR(D);
% Input :  p × ℓ polynomial matrix
D
% Output:  boolean variable obs, p×
ℓ polynomial matrix M
%
% Builds 0-degree rows generated by
the rows of M
% and checks if they have rank=ℓ; if
so, output variable obs=1.
% At each step the degree of one row
is decreased by adding a suitable
% polynomial combination of other rows.
This goes on until no lowering
% is possible, or the above condition
is satisfied.
% Output matrix M is the result of
such reduction.
%
%
   M = D
   M=Order(M); % order rows in decreasing
```

degree order

obs=(**rank**(M^0) == ℓ); % check if 0-degree rows already enough

p=rowdim(M);

i=p-rowdim(M^0); % otherwise start from first row of higher degree

while ((not obs) and ($i \geq 1$))

% Reduce until enough 0-degree rows or no more reduction possible

if ($\exists$ real n s.t. $M_{hc}(i,:) = nM_{hc}(i+1:p,:)$) then

% check if highest coefficient is linearly dependent

% from those of rows of lower or equal degree

h=**polann**(n, M); %polynomial vector to reduce degree

$m = M(i,:) - hM(i+1:p,:)$; % degree(m)< degree(M(i,:))

M=**Eliminate**(M, i); % eliminate row

if ($m \neq 0$) then

[M, j]=**Insert**(M, m);

% if new vector is not $\mathbf{0}$ insert it as j-th row

% so as to keep rows ordered

```
by degree
                    if (degree(m) == 0)
then
                         obs=(rank(M^0) ==
ℓ);
                         i = j-1; % determine
new row to examine
                    else i = j end
               else p = p-1;   i = i-1;
end
          else i = i - 1;
          end
     end
```

```
function ctr=CTRB(R);
%
% Input:  Polynomial matrix R with
p rows and q columns
% Output:  Boolean varaible ctr=1 if
ker(R(d/dt)) is controllable
%
%
   [P,ctr]=RPR(R^T);
    % Check if the 0-degree columns
generated by
    % the columns of R have rank p
   if (not ctr) then

      P = P^T;
      P=COLPRP(P); % Bring P in
column proper form
      if (rowdim(P)>coldim(P)) then
       % if the matrix was not of full
row rank

         [P,ctr]=RPR(P);
          %Check if the 0=degree rows
generated by
          % column proper form have
rank=q
      end

   end
```

REPRESENTATIONS

$$R(\tfrac{d}{dt})w = 0 \qquad w = M(\tfrac{d}{dt})\ell$$

$$R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell$$

$$E\tfrac{d}{dt}x + Fx + Gw = 0$$

$$\begin{gathered} \tfrac{d}{dt}x = Ax + Bu \\ y = Cx + Du \\ w \approx (u, y) \end{gathered}$$

$$\begin{gathered} P(\tfrac{d}{dt})y = Q(\tfrac{d}{dt})u \\ w \approx (u, y) \end{gathered}$$

$$\begin{gathered} y = G(s)u \\ w \approx (u, y) \end{gathered}$$

! Algorithms for passing among them ! Algorithms for testing various properties ! Algorithms for synthesis

SIMULATION

$$R(\frac{d}{dt})w = 0$$

will, of course, have many solutions, due to

- free variables among $(w_1, w_2, \cdots, \mathtt{w_m})$
- free initial conditions

In order to simulate a response, we need additional data

$$K(\frac{d}{dt})w = f \quad f : \mathbb{R} \to \mathbb{R}^{\bullet} \text{ given}$$
$$S(\frac{d}{dt})w(0) = a \quad a \in \mathbb{R}^{\bullet} \text{ given}$$

Does there exist a solution?

Does there exist a unique solution?

If so, algorithm

$$(R, K, S, f, a) \mapsto w$$

full plant behavior)

$\mathcal{P}_{\text{full}} = \{(v, c) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{v}+\text{c}}) \mid (v, c)$ satisfies the plan

plant behavior

$\mathcal{P} = \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{v}}) \mid \exists\, c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{c}})$ such that $(v, c$

controlled behavior $\mathcal{K}$ defined by

$\mathcal{K} = \{v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{v}}) \mid \exists\, c \in \mathcal{C}$ such that $(v, c) \in \mathcal{P}_{\text{full}}\}$.

For what $\mathcal{K} \in \mathfrak{L}^{\text{v}}$ does there exists a $\mathcal{C} \in \mathfrak{L}^{\text{c}}$ that implements $\mathcal{K}$?

$\mathcal{C}$ ***implements*** $\mathcal{K}$ if the above relation holds between $\mathcal{C}$ and $\mathcal{K}$.

For what $\mathcal{K} \in \mathfrak{L}^{\text{v}}$ does there exists a $\mathcal{C} \in \mathfrak{L}^{\text{c}}$ that implements $\mathcal{K}$?

Sheet

hidden behavior and is denoted as $\mathcal{N}$. It is formally defined as

$$\mathcal{N} = \{v \in \mathcal{P} \mid (v, 0) \in \mathcal{P}_{\text{full}}\}.$$

Theorem 1 (Controller implementability theorem) : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\text{v}+\text{c}}$ be the full plant behavior, $\mathcal{P} \in \mathfrak{L}^{\text{v}}$ the manifest plant behavior, and $\mathcal{N}$ the hidden behavior. Then $\mathcal{K} \in \mathfrak{L}^{\text{v}}$ is implementable by a controller $\mathcal{C} \in \mathfrak{L}^{\text{c}}$ acting on the control variables if and only if*

$$\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}.$$

Sheet Assume also that the transfer function $G_{(d,u)\mapsto(f,y)}$ associated with $\mathcal{P}_{\text{full}}$ has the following properties:

(i) $G_{(d,u)\mapsto(f,y)}$ is proper,

(ii) $G^{\infty}_{u\mapsto f}$ is injective,

(iii) $G^{\infty}_{d\mapsto y}$ is surjective, and

(iv) $G^{\infty}_{u\mapsto y} = 0$.

Let $\mathcal{N} \in \mathfrak{L}^{\mathrm{d+f}}$ be the hidden behavior, and $\mathcal{P} \in \mathfrak{L}^{\mathrm{d+f}}$ be the plant behavior associated with $\mathcal{P}_{\text{full}}$. Assume that the behavior $\mathcal{K} \in \mathfrak{L}^{\mathrm{d+f}}$ satisfies:

(v) $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, i.e., $\mathcal{K}$ is an implementable controlled behavior,

(vi) in $\mathcal{K}$, d is input and f is output, and

(vii) the transfer function $K_{d\mapsto f}$ from d to f in $\mathcal{K}$ is proper.

Then there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathrm{u+y}}$ such that

1. **$\mathcal{C}$ implements $\mathcal{K}$,**
2. **in $\mathcal{C}$, y is input and u is output, and**
3. **the transfer function $C_{y\mapsto u}$ from y to u in $\mathcal{C}$ is proper.**

SALIENT FEATURES

- Dynamical system = a behavior
  input/output structure: important special case
- First principles models $\Longrightarrow$ latent variables
  state variables: important special case
- Control = interconnection
  feedback : important special case
- Modelling complex systems = tearing & zooming
  input-to-output cascade and feedback: limited special case

terminal	equations
electrical	$V_1 = V_2, \quad I_1 + I_2 = 0$
1-D mechanical	$q_1 - q_2, \quad F_1 + F_2 = 0$
2-D mechanical	$x_1 = x_2,\ y_1 = y_2, \theta_1 = -\theta_2,$ $X_1 + X_2 = 0,\ Y_1 + Y_2 = 0,\ T$
thermal	$T_1 = T_2,\ Q_1 + Q_2 = 0$
fluidic	$p_1 = p_2,\ f_1 + f_2 = 0$
logical output $\rightarrow$ input	u=y
etc. etc.	etc. etc.

RLC-circuit

Problem: Model the relation between (V_1, I_1, V_2, I_2)

STABILITY

The system

$$R(\frac{d}{dt})w = 0$$

is *stable* if $(w \in \mathfrak{B}) \Rightarrow (w(t) \to 0$ for $t \to \infty$

Stable $\Leftrightarrow$
$(\lambda \in \mathbb{C}$ and $\mathrm{rank}(R(\lambda)) < \mathtt{w}) \Rightarrow (R_\varepsilon(\lambda) < 0)$

is *stabilizable* if for all $w \in \mathfrak{B}$ there is $w' \in \mathfrak{B}$ such that

$$w'(t) = w(t) \quad \text{for } t < 0$$
$$w'(t) \to 0 \quad \text{for } t \to \infty$$

Stabilizable $\Leftrightarrow$
$(\lambda \in \mathbb{C}$ and $\mathrm{rank}(R(\lambda)) < \mathrm{rank}(R)) \Rightarrow (R_\varepsilon(\lambda) < 0)$

STABILIZABILITY

A plant is stabilizable by a regular control interconnection if and only if

1. $\mathcal{N}$ **is stable**

2. $\mathcal{P}$ **is stabilizable**

Note : $\mathcal{N}$ **is stable = "detectability"**
i.e $c = 0 \Rightarrow w(t) \rightarrow 0$ **for** $t \rightarrow \infty$